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Best and Partial Best L_1 Approximations by Polynomials to Certain Rational Functions

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1.1. INTRODUCTION AND DEFINITION OF THE PROBLEM

In this paper we are concerned exclusively with approximating real-valued functions of a real variable by real polynomials, on the interval $[-1, 1]$.

Let $p_n(A, x) \equiv \sum_{i=0}^{n} a_i x^i \in P_n$ and $f(x) \in L\,[-1, 1]$ be the space of integrable functions on $[-1, 1]$. $p_n(A^*, x)$ is defined to be a best L_1 approximation from P_n to $f(x)$ on $[-1, 1]$ if

$$E_n^1(f) \equiv \int_{-1}^{1} |f(x) - p_n(A^*, x)|\,dx \leqslant \int_{-1}^{1} |f(x) - p_n(A, x)|\,dx$$

for all coefficient vectors A in Euclidean $n + 1$-space. $E_n^1(f)$ shall be referred to as the minimal, or best, L_1 deviation of f with respect to P_n.

We denote by $U_r(x)$, the Chebyshev polynomial of the second kind of degree r for all real r. By a U-polynomial of degree N, we mean an expression of the form $\sum_{j=0}^{N} e_j U_j(x)$, where $\{e_j\}_{j=0}^{N}$ are real scalars. We let $P_{m,n}$ denote the class of all U-polynomials of degree n where e_m is fixed and nonzero for a particular $m \leqslant n$.

It follows from the argument in [1, p. 10] that there exists a $p_{m,n}^* \in P_{m,n}$ such that for $f(x) \in L[-1, 1]$:

$$\tilde{E}_{m,n}^1(f) \equiv \int_{-1}^{1} |f(x) - p_{m,n}^*(x)|\,dx \leqslant \int_{-1}^{1} |f(x) - p_{m,n}(x)|\,dx$$

for all $p_{m,n} \in P_{m,n}$.

This $p_{m,n}^*$ is defined to be a partial best L_1 approximation to f from P_n. The motivation for investigating the partial minimum phenomenon in L_1 for this class of rational functions, stems from its analogue in uniform approximation. There, taking the Fourier expansion in Chebyshev polynomials of the first kind, Rivlin [5] has shown that the truncated series polynomial, suitably modified, is the best uniform approximation.

1.2. Best Approximation

Proposition. *For λ, μ real, $|\lambda| > 1$, $\mu \neq 0$, the functions*

$$\text{(i)} \quad g(x) = \frac{1}{\lambda - x}$$

$$\text{(iia)} \quad g(x) = \frac{1}{\lambda^2 - x^2} \qquad \text{(iib)} \quad g(x) = \frac{x}{\lambda^2 - x^2}$$

$$\text{(iiia)} \quad g(x) = \frac{1}{\mu^2 + x^2} \qquad \text{(iiib)} \quad g(x) = \frac{x}{\mu^2 + x^2}$$

all have their unique best L_1 approximations on $[-1, 1]$ from P_n given by the polynomial interpolating $g(x)$ at the roots of $U_{n+1}(x)$ (cf. [4, *Theorems* 4.3 *and* 4.4]).

Explicit expressions can be found for the L_1 deviation of these functions and their asymptotic behavior for k large is tabulated here:

$$\text{(i)} \quad E_k^1\left(\frac{1}{\lambda - x}\right) \sim 4(\lambda - (\lambda^2 - 1)^{1/2})^{k+2}$$

$$\text{(iia)} \quad E_{2k}^1\left(\frac{1}{\lambda^2 - x^2}\right) = E_{2k+1}^1\left(\frac{1}{\lambda^2 - x^2}\right) \sim \frac{4}{\lambda}(\lambda - (\lambda^2 - 1)^{1/2})^{2k+3}$$

$$\text{(iib)} \quad E_{2k+1}^1\left(\frac{x}{\lambda^2 - x^2}\right) = E_{2k+2}^1\left(\frac{x}{\lambda^2 - x^2}\right) \sim 4(\lambda - (\lambda^2 - 1)^{1/2})^{2k+4}$$

$$\text{(iiia)} \quad E_{2k}^1\left(\frac{1}{\mu^2 + x^2}\right) = E_{2k+1}^1\left(\frac{1}{\mu^2 + x^2}\right) \sim \frac{4}{\mu}((1 + \mu^2)^{1/2} - \mu)^{2k+3}$$

$$\text{(iiib)} \quad E_{2k+1}^1\left(\frac{x}{\mu^2 + x^2}\right) = E_{2k+2}^1\left(\frac{x}{\mu^2 + x^2}\right) \sim 4((1 + \mu^2)^{1/2} - \mu)^{2k+4}.$$

The case $g(x) = 1/(\lambda - x)$ has been treated in [1, Addenda, Sects. 31, 32] and the others may be similarly derived.

1.3. Some Lemmas

We first present a sufficient condition for partial best L_1 approximations, (c.f. [2, Corollary 1.5]).

Lemma 1. *Let $f(x) \in L[-1, 1]$. Then $p_{m,n}^*$* is a best L_1 approximation to f from $P_{m,n}$ if

$$\int_{-1}^{1} \operatorname{sign}(f(x) - p_{m,n}^*(x))\, U_j(x)\, dx = 0 \qquad j = 0, 1, \ldots, n; \qquad j \neq m.$$

In the case $n = m$, $f \in C[-1, 1]$, it is also true that $p^*_{m,n}$ is unique by extending the arguments in [4, Sect. 4.5].

DEFINITION (i). Let $\alpha_\nu[\nu = 1,..., m]$ be the real or complex-conjugate roots of the polynomial

$$\rho_m(x) = \prod_{\nu=1}^{m} \left(1 - \frac{x}{\alpha_\nu}\right) \qquad m \geqslant 0 \tag{1.1}$$

where $\rho_m(x)$ is positive in the interior of the interval $[-1, 1]$ but is allowed simple roots at one or both ends of the interval. If $m = 0$, we interpret this product as 1. $\rho_m(x)$ as expressed in (1.1) is defined to be in its canonical form.

We now introduce the mapping $x = (1/2)(v + (1/v))$.

The real variable x, $|x| \leqslant 1$ is then related to the complex value v by the equation

$$x = \frac{1}{2}\left(v + \frac{1}{v}\right) \qquad |v| = 1 \qquad \operatorname{Im} v \geqslant 0.$$

DEFINITION (ii). Define the complex constants c_ν by

$$c_\nu^2 - 2c_\nu\alpha_\nu + 1 = 0 \qquad |c_\nu| \leqslant 1 \qquad [\nu = 1,..., m]$$

Then

$$\alpha_\nu = \frac{1}{2}\left(c_\nu + \frac{1}{c_\nu}\right).$$

DEFINITION (iii). Define $H_m(v)$ to be the modified image under the mapping $x = (1/2)(v + (1/v))$ of the canonical polynomial $\rho_m(x)$ by

$$H_m(v) = \prod_{\nu=1}^{m} (v - c_\nu).$$

LEMMA 2. *With $\rho_m(x)$ defined as in Definition* (i) *and $H_m(v)$ defined as in Definition* (iii) *we have that for $n \geqslant m$*

$$U_n(x, \rho_m) \equiv K_{n+1,m} \left[v^{n+1-2m} \frac{H_m(v)}{H_m(1/v)} - v^{2m-n-1} \frac{H_m(1/v)}{H_m(v)}\right] \frac{\rho_m(x)}{v - (1/v)}$$

is a polynomial in x of degree n whose coefficient of x^n is equal to one provided

$$K_{n+1,m} = 2^{-n} \prod_{\nu=1}^{m} (1 + c_\nu^2).$$

Note that $H_m(1/v)\, H_m(v) = \prod_{\nu=1}^{m} (1 + c_\nu^2)\, \rho_m(x)$.

Lemma 2 is stated in [1, p. 251] and in [3, p. 37].

LEMMA 3. *For any $\rho_m(x)$ defined as before and $n \geqslant m$*

$$\min_{\{A_k\}} \int_{-1}^{1} \frac{| x^n + A_1 x^{n-1} + \cdots + A_n | \, dx}{\rho_m(x)} = \int_{-1}^{1} \frac{| U_n(x, \rho_m)| \, dx}{\rho_m(x)} = 2K_{n+1,m} .$$

The proof of Lemma 3 is to be found in [1, p. 251].

LEMMA 4. *Let $\gamma_a(x)$ be a polynomial in x of degree a defined by*

$$\gamma_a(x) = 1 + t^2 - 2t \cos[a \cos^{-1}(x)]$$

and $\rho_a(x)$ be its canonical form as defined in Definition (i). *Then $\gamma_a(x) = \prod_{v=1}^{a} (1 + c_v^2) \rho_a(x)$ where c_v are the appropriate constants defined in Definition* (ii). *Furthermore, $H_a(v)$, the modified image of $\gamma_a(x)$ under the mapping $x = (1/2)(v + (1/v))$ is given by*

$$H_a(v) = v^a - t.$$

The proof is omitted.

LEMMA 5. *For $\rho_a(x)$ defined as in Lemma 4, we have that sign $U_{m+a}(x, \rho_a)$ is orthogonal on $[-1, 1]$ to $U_j(x)$ $0 \leqslant j \leqslant m + a - 1$, $j \neq m$; for all non-negative integers m and a.*

Proof. With $x = \cos\theta$ and $v = e^{i\theta}$:

$$\operatorname{sign} U_{m+a}(x, \rho_a) = \frac{2}{\pi i} \sum_{r=0}^{\infty} \frac{1}{2r+1} \left[\left(v^{m-a+1} \frac{H_a(v)}{H_a(1/v)} \right)^{2r+1} - \left(v^{a-m-1} \frac{H_a(1/v)}{H_a(v)} \right)^{2r+1} \right]$$

(c.f. [1, p. 252]). Therefore

$$\int_{-1}^{1} \operatorname{sign} U_{m+a}(x, \rho_a) \, U_j(x) \, dx = \frac{2}{\pi i} \sum_{r=0}^{\infty} \frac{1}{2r+1} I_r$$

where

$$\begin{aligned} I_r = {} & \frac{1}{2i} \int_0^{\pi} [v^{(j+1)} - v^{-(j+1)}] \\ & \times \left[\left(v^{m-a+1} \frac{H_a(v)}{H_a(1/v)} \right)^{2r+1} - \left(v^{a-m-1} \frac{H_a(1/v)}{H_a(v)} \right)^{2r+1} \right] d\theta \\ = {} & \frac{1}{2i} \int_{|v|=1} v^{(j+1)+(m-a+1)(2r+1)} \left[\frac{H_a(v)}{H_a(1/v)} \right]^{2r+1} \frac{dv}{iv} \\ & - \frac{1}{2i} \int_{|v|=1} v^{(m-a+1)(2r+1)-(j+1)} \left[\frac{H_a(v)}{H_a(1/v)} \right]^{2r+1} \frac{dv}{iv} . \end{aligned}$$

Note $1/[H_a(1/v)] = v^a/[1 - v^a t]$ has poles at $1/c_v$ $| 1/c_v | \geqslant 1$.

By the theorem of residues, the first term of the last equation gives zero contribution for $r \geqslant 0$, $m \geqslant 0$, and all $j \geqslant 0$, whereas the second term gives zero contribution for $r \geqslant 1$, $m \geqslant 0$, $a \geqslant 0$, and $0 \leqslant j \leqslant 3m + 1$.

Now for $r = 0$, we consider the following cases for the second term.

(a) $j \leqslant m - 1$:

$$\int_{|v|=1} v^{m-j} \frac{(v^a - t)}{1 - v^a t} \frac{dv}{iv} = 0$$

by the theorem of residues.

(b) $j = m$:

$$\int_{|v|=1} \frac{(v^a - t)}{1 - v^a t} \frac{dv}{iv} \neq 0.$$

(c) $m < j \leqslant m + a - 1$ $(a > 1)$:

Set

$$\Phi_{m,a}(j) \equiv \int_{-\pi}^{\pi} e^{i(m-j)\theta} \frac{(e^{ia\theta} - t)}{1 - te^{ia\theta}} d\theta.$$

Make the change of variable $\theta = \phi + (2\pi/a)$.

$$\begin{aligned} \Phi_{m,a}(j) &= e^{i(m-j)(2\pi/a)} \int_{-\pi-(2\pi/a)}^{\pi-(2\pi/a)} e^{i(m-j)\phi} \frac{(e^{ia\phi} - t)}{1 - te^{ia\phi}} d\phi \\ &= e^{i(m-j)(2\pi/a)} \Phi_{m,a}(j) \end{aligned}$$

by periodicity. This is contradictory unless

$$\Phi_{m,a}(j) = 0 \qquad \text{for} \quad m + 1 \leqslant j \leqslant m + a - 1.$$

1.4. Partial Best Approximation

THEOREM. *Let a, b be non-negative integers $a > 0$ and*

$$f(x) = \sum_{j=0}^{\infty} t^j U_{aj+b}(x) \qquad |t| < 1 \tag{1.2}$$

Then

$$f(x) = \frac{U_b(x) - tU_{b-a}(x)}{1 + t^2 - 2t \cos a(\cos^{-1} x)}. \tag{1.3}$$

Furthermore, for $m = ak + b$ and $e_m = t^k/(1 - t^2)$:

$$p_{m,n}^*(x) = q_m(x) \equiv \sum_{j=0}^{k} t^j U_{aj+b}(x) + \frac{t^{k+2}}{1 - t^2} U_{ak+b}(x) \tag{1.4}$$

$$\tilde{E}_{m,n}^1(f) = \frac{2\,|\,t\,|^{k+1}}{1 - t^2} \tag{1.5}$$

for $m \leqslant n < m + a$.

Proof. (1.3) follows from (1.2) since

$$\sum_{j=0}^{\infty} t^j U_{aj+b}(x) = \frac{1}{\sin\theta} \operatorname{Im}\left[e^{i(b+1)\theta} \sum_{j=0}^{\infty} (te^{ia\theta})^j\right]$$

and the right-hand side is a convergent geometric series for $|\,t\,| < 1$. Therefore

$$f(x) = \frac{1}{\sin\theta} \frac{\sin(b+1)\theta - t\sin(b-a+1)\theta}{1 + t^2 - 2t\cos a\theta}$$

from which the result follows.

Let us first consider $n = m$. Set $\epsilon(x) = f(x) - q_m(x)$. Then putting $x = \cos\theta$, we obtain

$$\epsilon(x) = \frac{t^{k+1}}{(1 - t^2)} \frac{\begin{pmatrix}\sin[a(k+1) + b + 1]\theta - 2t\sin[ak + b + 1]\theta \\ + t^2 \sin[a(k-1) + b + 1]\theta\end{pmatrix}}{\sin\theta(1 + t^2 - 2t\cos a\theta)}. \tag{1.6}$$

Putting $\gamma_a(x) = 1 + t^2 - 2t\cos a\,(\cos^{-1} x)$:

$$N = a(k+1) + b \qquad \text{and} \qquad v = e^{i\theta}$$

$$\epsilon(x) = \frac{t^{k+1}}{1 - t^2} \frac{v^{(N+1)-2a}(v^a - t)^2 - v^{2a-(N+1)}(v^{-a} - t)^2}{\gamma_a(x)(v - (1/v))}.$$

Therefore, by Lemma 4 and the note on Lemma 2:

$$\epsilon(x) = \frac{t^{k+1}}{1 - t^2} \frac{v^{(N+1)-2a} H_a^2(v) - v^{2a-(N+1)} H_a^2(1/v)}{H_a(v)\, H_a(1/v)(v - (1/v))}.$$

Thus,

$$\int_{-1}^{1} |\,\epsilon(x)|\, dx = \frac{|\,t\,|^{k+1}}{1 - t^2} \frac{1}{K_{N+1,a}} \int_{-1}^{1} \left| \frac{U_N(x, \rho_a)}{\rho_a(x)} \right| dx = \frac{2\,|\,t\,|^{k+1}}{1 - t^2}$$

by Lemma 3.

Since the approximation of $f(x)$ by a polynomial from the class $P_{m,m}$ is equivalent here to the minimization (by norm) of a rational form whose numerator is a polynomial of degree $N = m + a$ with its highest coefficient prescribed and whose denominator is a prescribed polynomial of degree a in x, positive on the given interval, we have from Lemma 3 that $\epsilon(x)$ is minimal and $p^*_{m,m}(x) = q_m(x)$.

$p^*_{m,m}(x)$ is obviously unique, due to the determinateness of $U_N(x, \rho_a)$.

Now, sign $(f(x) - q_m(x)) = \pm$ sign $U_{m+a}(x, \rho_a)$ and from Lemma 5, sign $U_{m+a}(x, \rho_a)$ is orthogonal to $U_j(x)$, $0 \leqslant j \leqslant m + a - 1, j \neq m$. Hence, from Lemma 1, $q_m(x)$ is a partial best U-polynomial approximation in the L_1 norm to $f(x)$, among polynomials of degree $m + d$ for $0 \leqslant d \leqslant a - 1$ with $e_m = t^k/(1 - t^2)$. We now prove its uniqueness.

From (1.6) we have $\epsilon(x) = (t^{k+1}/(1 - t^2)) \sin((m + 1)\theta + \psi)$, where ψ is defined by

$$\sin \psi = \frac{(1 - t^2) \sin a\theta}{\gamma_a(\cos \theta)}$$

$$\cos \psi = \frac{-2t + (1 + t^2) \cos a\theta}{\gamma_a(\cos \theta)}. \tag{1.7}$$

From (1.7) we see that as θ varies from 0 to π, ψ increases from 0 to $a\pi$. Therefore $(m + 1)\theta + \psi$ increases continuously from 0 to $(m + a + 1)\pi$ as θ runs from 0 to π and $\epsilon(x)$ has $m + a$ alternations of sign, and hence, real single roots on $(-1, 1)$. Let the roots of $\epsilon(x)$ be α_i on $(-1, 1)$ for $i = 1,\ldots, m + a$. Suppose $p_{m,m+d}$ is another partial best L_1 approximation for $0 < d \leqslant a - 1$. Then by extending the argument in [4, Lemma 4.5], $f - p_{m,m+d}$ changes sign at the α_i. From this it would follow that $p_{m,m+d} - q_m$ has $m + a$ roots, which is clearly impossible.

COROLLARY 1. *If α, β are arbitrary real numbers; a, b are nonnegative-integers $a > 0$; $|t| < 1$, $m = ak + b$, and $m \leqslant n < m + a$, then $f(x) = \beta + \alpha \sum_{j=0}^{\infty} t^j U_{aj+b}(x)$ can be expressed as*

$$f(x) = \frac{\beta(1 + t^2) - 2\beta t \cos a(\cos^{-1} x) + \alpha U_b(x) - \alpha t U_{b-a}(x)}{1 + t^2 - 2t \cos a(\cos^{-1} x)} \tag{1.8}$$

and

$$p^*_{m,n}(x) = \beta + \alpha \sum_{j=0}^{k} t^j U_{aj+b}(x) + \frac{\alpha t^{k+2}}{1 - t^2} U_{ak+b}(x)$$

$$E_n^1(f) \leqslant \tilde{E}^1_{m,n}(f) = \frac{2\,|\alpha|\,|t|^{k+1}}{1 - t^2}.$$

COROLLARY 2. *For* $\sigma = 1,\ldots, a$ *the best* L_1 *approximation from* $P_{m-\sigma}$ *to* $f(x) - (\alpha t^k/(1-t^2))\, U_m(x)$ *is* $\beta + \alpha \sum_{j=0}^{k-1} t^j U_{aj+b}(x)$ *and*

$$E^1_{m-\sigma}\left(f - \frac{\alpha t^k U_m}{1-t^2}\right) = \frac{2\,|\,\alpha\,|\,|\,t\,|^{k+1}}{1-t^2} \qquad \sigma = 1,\ldots, a.$$

EXAMPLE (i).

$$g(x) = \frac{1}{x-\lambda}, \qquad \lambda > 1.$$

Choose $t = \lambda - (\lambda^2 - 1)^{1/2}$, then $|\,t\,| < 1$. Choose $a = 1$, $b = 0$, $\beta = 0$, $\alpha = -2t$. Then (1.8) becomes $1/(x-\lambda)$. Thus,

$$p^*_{n,n}(x) = -2t \sum_{j=0}^{n-1} t^j U_j(x) - \frac{2t^{n+1}}{1-t^2}\, U_n(x)$$

$$\tilde{E}^1_{n,n}\left(\frac{1}{x-\lambda}\right) = \frac{4\,|\,t\,|^{n+2}}{1-t^2}.$$

EXAMPLE (iia).

$$g(x) = \frac{1}{x^2-\lambda^2}, \qquad |\,\lambda\,| > 1.$$

Choose $t = -(1-2\lambda^2) - 2\lambda(-1+\lambda^2)^{1/2}$. Then $0 < t < 1$. Choose $a = 2$, $b = 0$, $\beta = 0$, $\alpha = -4t/(1+t)$. Then (1.8) becomes $1/(x^2 - \lambda^2)$ and

$$\tilde{E}^1_{2k,2k}\left(\frac{1}{x^2-\lambda^2}\right) = \tilde{E}^1_{2k,2k+1}\left(\frac{1}{x^2-\lambda^2}\right) = \frac{8\,|\,t\,|^{k+2}}{(1+t)(1-t^2)}.$$

EXAMPLE (iib).

$$g(x) = \frac{x}{x^2-\lambda^2}, \qquad |\,\lambda\,| > 1.$$

With the same choice of t as in (iia) choose $a = 2$, $b = 1$, $\beta = 0$, $\alpha = -2t$. Then

$$\tilde{E}^1_{2k+1,2k+1}\left(\frac{x}{x^2-\lambda^2}\right) = \tilde{E}^1_{2k+1,2k+2}\left(\frac{x}{x^2-\lambda^2}\right) = \frac{4\,|\,t\,|^{k+2}}{1-t^2}.$$

EXAMPLE (iiia).

$$g(x) = \frac{1}{\mu^2+x^2}, \qquad |\,\mu\,| > 0.$$

Choose $t = -(1+2\mu^2) + 2\mu(1+\mu^2)^{1/2}$. Then $-1 < t < 0$. Choose $a = 2$, $b = 0$, $\beta = 0$, $\alpha = -4t/(1+t)$. Then (1.8) becomes $1/(\mu^2 + x^2)$ and

$$\tilde{E}^1_{2k,2k}\left(\frac{1}{\mu^2+x^2}\right) = \tilde{E}^1_{2k,2k+1}\left(\frac{1}{\mu^2+x^2}\right) = \frac{8\,|\,t\,|^{k+2}}{(1-|\,t\,|)(1-t^2)}.$$

EXAMPLE (iiib).

$$g(x) = \frac{x}{\mu^2 + x^2}, \qquad |\mu| > 0.$$

With the same choice of t as in (iiia) choose $a = 2, b = 1, \alpha = 0, \beta = -2t$. Then

$$\tilde{E}^1_{2k+1,2k+1}\left(\frac{1}{\mu^2 + x^2}\right) = \tilde{E}^1_{2k+1,2k+2}\left(\frac{x}{\mu^2 + x^2}\right) = \frac{4\,|\,t\,|^{k+2}}{1 - t^2}.$$

1.5. CONCLUSION

The partial best L_1 approximations described above possess the advantage that their coefficients are readily available. Furthermore, one may show for the rational functions considered, that if the proximity of the partial best L_1 approximation is expressed as the ratio of $\tilde{E}^1_{m,n}(g)$ to $E_n{}^1(g)$ then, in the limit, this is determined by the *a priori* factor $1/(1 - t^2)$.

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REFERENCES

1. N. I. AKHIEZER, "Theory of Approximation," Ungar, New York, 1956.
2. B. R. KRIPKE AND T. J. RIVLIN, Approximation in the Metric of $L^1(x, \mu)$, *Trans. Amer. Math. Soc.* **119** (1965), 101–122.
3. G. MEINARDUS, "Approximation of Functions: Theory and Numerical Methods," Springer-Verlag, New York, 1967.
4. J. RICE, "Approximation of Functions," Vol. 1, Addison-Wesley, Reading, Mass., 1964.
5. T. J. RIVLIN, Polynomials of best uniform approximation, *Numer. Math.* **4** (1962), 345–349.